NORTH-HOLLAND

On Biclique Decompositions of Complete t-partite Graphs

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Dedicated to Professor John Maybee on the occasion of his 65th birthday.

Submitted by J. Richard Lundgren

ABSTRACT

We use a stronger version of the Graham-Pollack theorem for biclique decompositions of graphs to establish a correspondence between exact biclique decompositions of complete t-partite graphs and exact biclique decompositions of the complete graph on t vertices. In addition, we obtain some results for exact biclique decompositions of complete graphs.

1. INTRODUCTION

Let G be a graph with n vertices. A *biclique* of G is an edge subgraph of G which is a complete bipartite graph. If R and S are disjoint subsets of the vertices of G, then $B(R, S)$ denotes the biclique consisting of those edges $\{r, s\}$ where $r \in R$ and $s \in S$. We say that $B(R, S)$ *contains* the vertex v if $v \in R \cup S$. The bicliques

$$B(R_1, S_1), B(R_2, S_2), \ldots, B(R_m, S_m) \tag{1}$$

of G are a *biclique decomposition* of G provided each edge of G belongs to exactly one of the bicliques. Clearly, G has a biclique decomposition with

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$n-1$ or fewer bicliques. The *biclique decomposition number*, $\mathrm{bd}(G)$, of G is the smallest number of bicliques among all biclique decompositions of G.

Let A be an adjacency matrix of G, and let n_+ and n_- denote the numbers of positive and negative eigenvalues of A, respectively, including multiplicities. The well-known theorem of Graham and Pollak [5] (see also [8, 12, 9, 10]) asserts that the number of bicliques in any biclique decomposition of G is at least the maximum of n_+ and n_-, that is, $\mathrm{bd}(G) \geq \max\{n_+, n_-\}$. If $\mathrm{bd}(G) = \max\{n_+, n_-\}$, then G is an *eigensharp graph.* Eigensharp graphs have been studied in [6]. If the number of bicliques in a biclique decomposition of a graph G equals $\mathrm{bd}(G)$, then the decomposition is an *exact* biclique decomposition of G. Since the complete graph K_n with n vertices has $n-1$ negative eigenvalues, K_n is an eigensharp graph and any biclique decomposition of K_n has at least $n-1$ bicliques. Exact biclique decompositions of K_n have been studied in [1, 3, 2, 4, 7, 11, 13]. In Section 2, we discuss the relationship between biclique decompositions of G and certain matrix decompositions of A, and then examine the proof of the Graham-Pollak theorem more closely in order to obtain a slightly more general result.

If u and v are vertices of G such that a vertex w of G is adjacent to u if and only if w is adjacent to v, then we say that u and v *have the same neighbors in G.* Assume that u and v have the same neighbors in G. Let G' be the graph obtained from G by removing vertex u. Since induced subgraphs of complete bipartite graphs are complete bipartite graphs, it follows that $\mathrm{bd}(G') \leq \mathrm{bd}(G)$. If (1) is a biclique decomposition of G', then since u and v have the same neighbors, a biclique decomposition of G can be obtained by adjoining u to each R_i and each S_i which contains v. Thus, $\mathrm{bd}(G) \leq \mathrm{bd}(G')$. Therefore, the biclique decomposition numbers of G and G' are equal, and there exists an exact biclique decomposition of G such that each biclique containing u or v contains both u and v. It is also easy to verify that G and G' have the same number of positive eigenvalues and the same number of negative eigenvalues. Hence, in particular, G is eigensharp if and only if G' is.

It is not necessarily the case that in an exact biclique decomposition of G a biclique containing one of the vertices u and v contains both u and v. For example, let G be the graph with vertices $1, 2, \ldots, 10$ whose edges are the edges of the cycle

$$1\text{—}2\text{—}3\text{—}4\text{—}5\text{—}6\text{—}7\text{—}8\text{—}1$$

and the edges joining vertices 9 and 10 to vertices 1, 3, 5 and 7.[1] Note that vertices 9 and 10 have the same neighbors in G. Since H contains an

[1] We are indebted to Kevin Vander Meulen for this example.

induced cycle of length 8, and since the biclique decomposition number of a cycle of length 8 is clearly 4, the biclique decomposition number of G is at least 4. The bicliques

$$B(\{5,7\},\{6,10\}), B(\{3,5\},\{4,9\}), B(\{1,3\},\{2,10\}), B(\{1,7\},\{8,9\})$$

are an exact biclique decomposition of G, and the first biclique contains vertex 10 but not vertex 9. However, in Section 3 we show that if u and v have the same neighbors in a graph, and the graph has an exact biclique decomposition which has a biclique which contains exactly one of u and v, then the graph is not an eigensharp graph. Equivalently, if G is an eigensharp graph and if u and v are distinct vertices having the same neighbors in G, then for any exact biclique decomposition (1) of G we have $u \in R_i$ if and only if $v \in R_i (i = 1, 2, \ldots, m)$, and $u \in S_i$ if and only if $v \in S_i (i = 1, 2, \ldots, m)$.

We use this result to study exact biclique decompositions of eigensharp graphs which contain pairs of vertices having the same neighbors. In particular, we extend many of the results for exact biclique decompositions of complete graphs to exact biclique decompositions of complete t-partite graphs. Throughout let $K_{n_1,n_2,\ldots,n_t}$ denote the complete t-partite graph whose vertices are partitioned into sets $V_1, V_2, \ldots, V_t$ of cardinalities $n_1, n_2, \ldots, n_t$, respectively, with an edge joining every pair of vertices from distinct sets of the partition. The sets $V_1, V_2, \ldots, V_t$ are the *partite sets* of $K_{n_1,n_2,\ldots,n_t}$. We always assume that each of the parts is nonempty. Since any two vertices in the same part of a complete t-partite graph have the same neighbors, it follows that a complete t-partite graph is eigensharp and has biclique decomposition number equal to $t-1$. Let K_t denote the complete graph with vertices $1, 2, \ldots, t$. If

$$B(R_1, S_1), B(R_2, S_2), \ldots, B(R_{t-1}, S_{t-1})$$

is a biclique decomposition of K_t, then clearly

$$B\left(\bigcup_{i\in R_1} V_i, \bigcup_{j\in S_1} V_j\right), B\left(\bigcup_{i\in R_2} V_i, \bigcup_{j\in S_2} V_j\right), \ldots, B\left(\bigcup_{i\in R_{t-1}} V_i, \bigcup_{j\in S_{t-1}} V_j\right) \tag{2}$$

is an exact biclique decomposition of $K_{n_1,n_2,\ldots,n_t}$. We show that the converse holds, namely that every exact biclique decomposition of $K_{n_1,n_2,\ldots,n_t}$ has the form (2) for some exact biclique decomposition (1) of K_t. We also characterize the exact biclique decompositions of the t-partite graph $K_{n,n,\ldots,n}$ in which the bicliques are all isomorphic. This generalizes the characterization of exact isomorphic biclique decompositions of complete graphs given in [2].

In Section 4, we apply the results of Section 3 to exact biclique decompositions of complete graphs.

2. GRAHAM-POLLAK REVISITED

Let G be a graph with vertices $1, 2, \ldots, n$. Let $A = [a_{ij}]$ be the (0, 1) adjacency matrix of G, where

$$a_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \text{ is an edge of G,} \\ 0 & \text{otherwise.} \end{cases}$$

If P is a subset of $\{1, 2, \ldots, n\}$, then the *characteristic vector* $\vec{P}$ of the set P is the 1 by n $(0, 1)$ vector with a 1 in position i if and only if $i \in P$. Suppose that (1) is a biclique decomposition of G. Let X be the m by n $(0, 1)$ matrix whose ith row is the vector $\vec{R}_i$, and let Y be the m by n $(0, 1)$ matrix whose ith row is the vector $\vec{S}_i$. Since $\vec{S}_i^T \vec{R}_i + \vec{R}_i^T \vec{S}_i$ is the adjacency matrix of the spanning subgraph $B(R_i, S_i)$ of $G\,(i = 1, 2, \ldots, m)$, it follows that $Y^T X$ and $X^T Y$ are (0, 1) matrices such that

$$Y^T X + X^T Y = \sum_{i=1}^{m} \vec{S}_i^T \vec{R}_i + \vec{R}_i^T \vec{S}_i = A.$$

Thus, for each biclique decomposition of G with m bicliques, there exist m by n $(0, 1)$ matrices X and Y such that $Y^T X + X^T Y = A$. Conversely, if $X = [x_{ij}]$ and $Y = [y_{ij}]$ are m by n $(0, 1)$ matrices such that $Y^T X + X^T Y = A$, then

$$B(\{k : x_{1k} = 1\}, \{k : y_{1k} = 1\}), \ldots, B(\{k : x_{mk} = 1\}, \{k : y_{mk} = 1\})$$

is a biclique decomposition of G. Thus, biclique decompositions of G and decompositions of A as $A = Y^T X + XY^T$ where X and Y are $(0, 1)$ matrices are essentially equivalent.

We now state a slightly stronger version of the Graham-Pollak theorem which is an immediate consequence of the relationship between biclique decompositions and (0, 1) matrices and the following lemma about real matrices. This lemma is essentially Lemma 7.4 in [8], and we include a proof for completeness.

LEMMA 1. *Let A be a real symmetric matrix of order n such that the numbers of positive, negative, and zero eigenvalues of A (including multiplicity) are n_+, n_-, and n_0, respectively. Let X and Y be m by n real* *matrices such that $A = Y^T X + X^T Y$. Then the rank of $Y^T X$, and hence of Y and X, is at least the maximum of n_+ and n_-. If $m = \max\{n_+, n_-\}$, then the row space of X equals the row space of $Y^T X$, and the row space of Y equals the row space of $X^T Y$.*

Proof. Let R^n denote the vector space of all n by 1 real vectors. Since A is a real, symmetric matrix, each of its eigenvalues is real, and there exists an orthogonal basis of R^n consisting of eigenvectors of A. Let V_+ denote the subspace of R^n spanned by those basis vectors corresponding to the positive eigenvalues of A. Then the restriction of A to the subspace V_+ is a positive definite matrix, that is, for $v \in V_+$, we have $v^T A v \geq 0$ with equality if and only if $v = 0$. Let

$$W = \{v : Y^T X v = 0\}$$

be the nullspace of $Y^T X$. Suppose that $v \in V_+ \cap W$. Then

$$0 = v^T (Y^T X + X^T Y) v = v^T A v,$$

and it follows that $v = 0$. Hence $V_+ \oplus W$ is a subspace of R^n. Since the dimension of W equals $n - \operatorname{rank}(Y^T X)$, we conclude that the rank of $Y^T X$ is at least n_+. A similar argument shows that the rank of $Y^T X$ is at least n_-.

Now suppose that $m = \max\{n_+, n_-\}$. Then the rank of $Y^T X$ equals m. Since the row space of $Y^T X$ is contained in the row space of X, and since the dimension of the row space of X is at most m, it follows that the row space of X and the row space of $Y^T X$ are the same. Similarly, the row space of Y and the row space of $X^T Y$ are the same. ∎

THEOREM 2. *Let G be a graph with vertices $1, 2, \ldots, n$. Let A be the adjacency matrix of G, and let n_+, n_-, and n_0 be the numbers of positive, negative, and zero eigenvalues of A, respectively, including multiplicity. Let* (1) *be a biclique decomposition of G, and let X and Y be the m by n matrices whose rows are the characteristic vectors of the R_i and S_i, respectively. Then $m \geq \max\{n_+, n_-\}$, and if equality holds, then the rows X are linearly independent and the rows of Y are linearly independent.*

3. EXACT BICLIQUE DECOMPOSITIONS OF t-PARTITE GRAPHS

We begin by studying exact biclique decompositions of eigensharp graphs which contain vertices with the same neighbors.

THEOREM 3. *Let G be an eigensharp graph with vertices $1, 2, \ldots, n$, and let $B(R_1, S_1), B(R_2, S_2), \ldots, B(R_m, S_m)$ be an exact biclique decomposition of G. Assume that vertices i and j have the same neighbors in G. Then for each $k = 1, 2, \ldots, m$, one has $i \in R_k$ if and only if $j \in R_k$, and $i \in S_k$ if and only if $j \in S_k$.*

Proof. Since i and j are necessarily nonadjacent, we may assume without loss of generality that neither i nor j belongs to S_k $(k = 1, 2, \ldots, m)$.

Let X and Y be the m by $n\,(0,1)$ matrices whose kth rows are the characteristic vectors of R_k and S_k, respectively. Let u be the m by $1\,(0,1)$ vector whose kth entry is a 1 if and only if i belongs to R_k. Similarly, let v be the m by $1\,(0,1)$ vector whose kth entry is a 1 if and only if j belongs to R_k. Since $B(R_1, S_1), B(R_2, S_2), \ldots, B(R_m, S_m)$ is a biclique decomposition of G, and since no S_k contains i, the vector u^TY is the characteristic vector of the set $\{w : w \text{ is adjacent to } u \text{ in } G\}$. Similarly, v^TY is the characteristic vector of the set $\{w : w \text{ is adjacent of } v \text{ in } G\}$. Since u and v have the same neighbors, $u^TY = v^TY$, and hence $(u^T - v^T)Y = 0$. By Theorem 2 the rows of Y are linearly independent and thus $u = v$. We conclude that for $k = 1, 2, \ldots, m$, R_k contains u if and only if R_k contains v. ∎

We have already seen that exact biclique decompositions of noneigensharp graphs need not satisfy the conclusions of Theorem 3. This is also the case for nonexact biclique decompositions. For example, let (1) be an exact biclique decomposition of a graph G, and let u and v be vertices with the same neighbors. Suppose that both u and v belong to R_1. Then

$$B(\{u\}, S_1), B(R_1 - \{u\}, S_1), B(R_2, S_2), B(R_3, S_3), \ldots, B(R_m, S_m)$$

is a (nonexact) biclique decomposition of G for which the conclusion of Theorem 3 does not hold.

Recall that the biclique decomposition number of a complete t-partite graph is $t - 1$. We now show that the only exact biclique decompositions of a complete t-partite graph are those arising from exact biclique partitions of K_t.

COROLLARY 4. *Let $K_{n_1, n_2, \ldots, n_t}$ be the complete t-partite graph whose partite sets are the nonempty sets $V_1, V_2, \ldots, V_t$, and let K_t be the complete graph with vertices $1, 2, \ldots, t$. If $B(R_1, S_1), B(R_2, S_2), \ldots, B(R_{t-1}, S_{t-1})$ is an exact biclique decompositions of $K_{n_1, n_2, \ldots, n_t}$, then each R_i are each S_i is a union of a subcollection of $V_1, V_2, \ldots, V_t$, and*

$$B(\{i : V_i \subseteq R_j\}, \{i : V_i \subseteq S_j\}) \qquad (j = 1, 2, \ldots, t - 1) \tag{3}$$

is an exact biclique decomposition of K_t*. Conversely, if*

$$B(X_1, Y_1), B(X_2, Y_2), \ldots, B(X_{t-1}, Y_{t-1})$$

is an exact biclique decomposition of K_t*, then*

$$B\left(\bigcup_{i\in X_1} V_i, \bigcup_{i\in Y_1} V_i\right), B\left(\bigcup_{i\in X_2} V_i, \bigcup_{i\in Y_2} V_i\right), \ldots, B\left(\bigcup_{i\in X_1} V_i, \bigcup_{i\in Y_1} V_i\right)$$

is an exact biclique decomposition of $K_{n_1,n_2,\ldots,n_t}$.

Proof. Suppose that $B(R_1, S_1), B(R_2, S_2), \ldots, B(R_{t-1}, S_{t-1})$ is an exact biclique decomposition of $K_{n_1,n_2,\ldots,n_t}$. Since the vertices in each V_j ($j = 1, 2, \ldots, t$) have the same neighbors in $K_{n_1,n_2,\ldots,n_t}$, it follows from Theorem 3 that each R_i and each S_i is a union of a subcollection of $V_1, V_2, \ldots, V_t$. Since $B(R_1, S_1), B(R_2, S_2), \ldots, B(R_{t-1}, S_{t-1})$ is an exact biclique decomposition of $K_{n_1,n_2,\ldots,n_t}$, (3) is an exact biclique decomposition of K_t. The converse is clear. ■

In [2] exact biclique decompositions

$$B(R_1, S_1), B(R_2, S_2), \ldots, B(R_{t-1}, S_{t-1}) \tag{4}$$

of K_t in which each R_i has the same number a of elements and each S_i has the same number b of elements were studied. Such biclique decomposition are called *exact isomorphic biclique decompositions* of K_t. It was shown that if K_t has an exact isomorphic biclique decomposition, say with $a \le b$, then $a = 1$ and $t = 2b$. In addition, exact isomorphic biclique decompositions of K_t were shown to correspond to regular tournaments with $t - 1$ vertices. Corollary 4 now implies the following characterization of exact isomorphic biclique decompositions of the complete t-partite graphs $K_{n,n,\ldots,n}$.

COROLLARY 5. *Let* $B(R_1, S_1), B(R_2, S_2), \ldots, B(R_{t-1}, S_{t-1})$ *be an exact biclique decomposition of the complete t-partite graph* $K_{n,n,\ldots,n}$ *such that each* R_i *has* a *elements, and each* S_j *has* b *elements, and* $a \le b$*. Then* (3) *is an exact isomorphic biclique decomposition of* K_t, $a = n$, n *divides* b, *and* $t = 2(b/n)$.

4. APPLICATIONS TO EXACT BICLIQUE DECOMPOSITIONS OF K_t

In this section we apply the results about biclique decompositions of t-partite graphs to certain types of biclique decompositions of complete graphs.

THEOREM 6. *Let $B(R_1,S_1), B(R_2,S_2), \ldots, B(R_{t-1},S_{t-1})$ be a biclique decomposition of K_t. Suppose the edges of $B(R_1,S_1)$, $B(R_2,S_2), \ldots, B(R_{s-1}, S_{s-1})$ form a complete subgraph H of K_t. Then H has s vertices, and if $i \geq s$ and $B(R_i,S_i)$ contains a vertex of H, then either each vertex of H is a vertex of R_i or each vertex of H is a vertex of S_i.*

Proof. Let x be the number of vertices of H. Since every biclique decomposition of H has at least $x-1$ bicliques, $x \leq s$. Let G be the edge subgraph obtained from K_t by removing the edges of H. Then G is a complete $(t-x+1)$-partite graph $K_{1,1,\ldots,1,x}$, and $B(R_s,S_s), B(R_{s+1},S_{s+1}), \ldots, B(R_{t-1},S_{t-1})$ is a biclique decomposition of G. It follows that $t-s-1 \geq t-x-1$. Therefore $x=s$. The theorem now follows from Corollary 4. ∎

The following result about exact biclique decomposition of complete graphs in which one of the bicliques is an edge is an immediate consequence of Theorem 6.

COROLLARY 7. *Let $B(R_1,S_1)$, $B(R_2,S_2), \ldots, B(R_{t-1},S_{t-1})$ be a biclique decomposition of K_t such that $R_1 = \{1\}$ and $S_1 = \{2\}$. Then if $i \geq 2$ and $B(R_i,S_i)$ contains either 1 or 2, then either both 1 and 2 are contained in R_i or both 1 and 2 are contained in S_i.*

Let n and t be positive integers. We now study the special exact biclique decompositions of K_{tn} in which t of the bicliques are isomorphic to the complete bipartite graph $K_{1,j}$ for $j = 1,2,\ldots,n-1$. First we give an example of such a bipartite decomposition. Let $B(R_1,S_1), B(R_2,S_2), \ldots, B(R_{t-1},S_{t-1})$ be an exact biclique decomposition of the complete t-partite graph $K_{n,n,\ldots,n}$. Assume without loss of generality that the partite sets of K are $\{1,2,\ldots,n\}$, $\{n+1,n+2,\ldots,2n\}, \ldots, \{tn-n+1, tn-n+2, \ldots, tn\}$. Then it follows that the bicliques $B(R_1,S_1), B(R_2,S_2), \ldots, B(R_{t-1},S_{t-1})$ along with the bicliques

$$B(\{jn+k\}, \{jn+1, jn+2, \ldots, jn+k-1\})$$
$$(j = 0,1,\ldots,t-1, \quad k = 2,3,\ldots,n)$$

form an exact biclique decomposition of K_{tn}. We now show that every special exact biclique decompositions of K_{tn} can be constructed in the above manner.

LEMMA 8. *Let $B(R_1,S_1)$, $B(R_2,S_2), \ldots, B(R_{tn-1},S_{tn-1})$ be a biclique* *decomposition of K_{tn} such that for $j = 0, 1, \ldots, t-1$ and $i = 1, 2, \ldots, n-1$, R_{jn+1} has one vertex and S_{jn+i} has i vertices. Then $B(R_n, S_n)$, $B(R_{2n}, S_{2n})$, $\ldots$, $B(R_{(t-1)n}, S_{(t-1)n})$ is an exact biclique decomposition of a complete t-partite graph $K_{n,n,\ldots,n}$.*

Proof. If k and k' are distinct integers such that $R_k = R_{k'}$, then replacing the bicliques $B(R_k, S_k)$ and $B(R_{k'}, S_{k'})$ by the biclique $B(R_k, S_k \cup S_{k'})$ gives a biclique decomposition of K_{tn} with fewer than $tn - 1$ biclique. Hence, we conclude that the singleton sets R_{jk+i} $(j = 0, 1, \ldots, t-1,\ i = 1, 2, \ldots, n-1)$ are distinct. By a similar argument, we conclude that the tn singleton sets R_{jn+i}, S_{jn+1} $(j = 0, 1, 2, \ldots, t-1,\ i = 1, 2, 3, \ldots, n-1)$ are disjoint. It follows that the vertices of K_{tn} are the vertices in the union of these tn sets.

For $l = 1, \ldots, n-1$ let G_l be the spanning subgraph of K_{tn} whose edges are the union of the edges in the bicliques $B(R_{jn+l}, S_{jn+l})$ $(j = 0, 1, \ldots, t-1,\ i = 1, 2, \ldots, l)$. We show by induction on l that G_l is the vertex disjoint union of $t(n-l-1)$ isolated vertices and t complete graphs each with $l+1$ vertices. That G_1 is the disjoint union of $t(n-2)$ isolated vertices and t edges follows from the preceding paragraph. Now suppose that $l > 2$. By induction G_{l-1} is the disjoint union of $t(n-l)$ isolated vertices and tK_l's. Let $W_1, W_2, \ldots, W_t$ be the vertices of these t K_l's. It follows that the bicliques $B(R_{jn+i}, S_{jn+i})$ $(j = 0, 1, \ldots, t-1,\ i = l, l+2, \ldots, n-1)$ along with the bicliques $B(R_n, S_n)$, $B(R_{2n}, S_{2n}), \ldots,$ $B(R_{(t-1)n}, S_{(t-1)n})$ are an exact biclique decomposition of a complete $t(n-l+1)$-partite graph. Consider a biclique $B(R_{jn+l}, S_{jn+l})$ where $0 \leq j \leq t-1$. If S_{jn+l} does not contain a vertex in $\bigcup_{k=1}^{t} W_k$, then replacing each biclique $B(\{r\}, S)$ such that $r \in S_j$ with the biclique $B(\{r\}, S_{n+l} \cup R_{jn+l})$ and deleting the biclique $B(R_{jn+l}, S_{jn+l})$ results in a biclique decomposition of K_{tn} with fewer than $tn - 1$ bicliques. Hence S_{jn+l} contains some vertex in $\bigcup_{k=1}^{t} W_k$. Since S_{jn+l} has exactly l vertices, it follows from Corollary 4 that $S_{jn+l} = W_k$ for some k. If $B(R'_{jn+l}, S'_{jn+l})$ is another biclique, then $S_{jn+l} \neq S'_{jn+l}$. For otherwise, replacing the bicliques $B(R_{jn+l}, S_{jn+l})$ and $B(R'_{jn+l}, S'_{jn+l})$ with $B(R'_{jn+l} \cup R'_{jn+l}, S_{jn+l})$ results in a biclique decomposition of K_{tn} with fewer than $tn-1$ bicliques. Hence G_l is a disjoint union of $t(n-l-1)$ isolated vertices and t K_{l+1}'s, and the lemma follows by induction. ■

The following is now an immediate consequence of Lemma 8 and Corollary 5.

THEOREM 9. *Let $B(R_1, S_1)$, $B(R_2, S_2), \ldots, B(R_{tn-1}, S_{tn-1})$ be a biclique decomposition of K_{tn} such that for $j = 0, 1, \ldots, t-1$ and $i = 1, 2, \ldots, n-1$, R_{jn+i} has one vertex and S_{jn+i} has i vertices. Then there* *exists an exact biclique decomposition* $B(X_1, Y_1)$, $B(X_2, Y_2), \ldots$, $B(X_{t-1}, Y_{t-1})$ *of the complete graph with vertices* $1, 2, \ldots, t$ *such that after relabeling* $R_{jn+i} = \{jn + i + 1\}$ *and* $S_{jn+i} = \{jn + 1, jn + 2, \ldots, jn + i\}$, *one has* $R_{jn} = \bigcup_{k \in X_j} V_k$ *for* $j = 0, 1, \ldots t - 1$, $i = 1, 2, \ldots, n - 1$, *and* $S_{jn} = \bigcup_{k \in Y_j} V_k$ *for* $j = 1, \ldots, t - 1$, $i = 1, 2, \ldots, n - 1$, *where* $V_j = \{(j - 1)n + 1, (j - 1)n + 2, \ldots, jn\}$ $(j = 1, 2, \ldots, t)$.

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